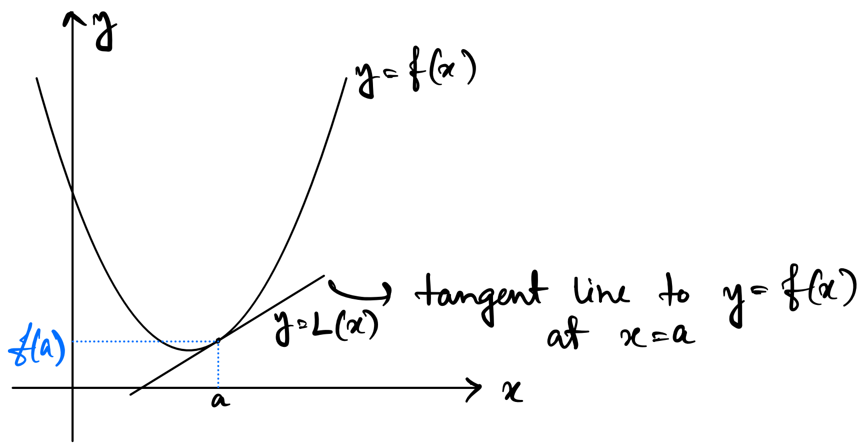
Linear Approximations & Differentials:



⊛ **Note**: Tangent is the best straight line approximation to $y = f(x)$ at $x = a$.

Now the equation of the tangent line passing through $(a, f(a))$ with slope $f'(a)$ is given by

$$y - f(a) = f'(a)(x - a)$$

$$\Rightarrow y = \underbrace{f(a) + f'(a)(x-a)}_{L(x),}$$

the linear approximation.

Eg. (a) Find the linear approximation of $f(x) = \sqrt[3]{x}$ at $a = 8$

(b) Use linear approximation to estimate $\sqrt[3]{8.05}$ & $\sqrt[3]{25}$

Sol^n:- (a) As like finding equation of tangent, we need to find

① $f(8)$ in this case 8

② $f'(8)$ too.

Now, $f(8) = \sqrt[3]{8} = 2$

& $f'(x) = \left[\sqrt[3]{x}\right]'$

$= \left(x^{1/3}\right)' = \frac{1}{3} x^{-2/3}$

So, $f'(8) = \frac{1}{3}(8)^{-2/3}$

$= \frac{1}{3}\left(2^3\right)^{-2/3}$

$= \frac{1}{3}\left(2^{-2}\right) = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$

Therefore, the linear approximation is:

$$L(x) = f(8) + f'(8)\cdot(x-8)$$

$$= 2 + \frac{1}{12}(x-8).$$

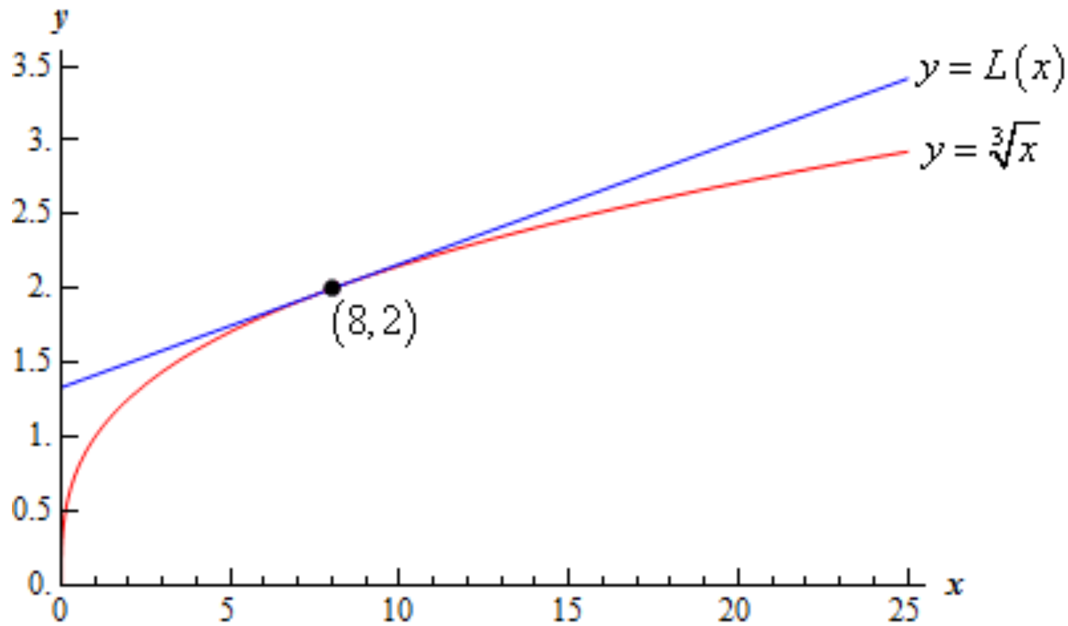
(b) To estimate $\sqrt[3]{8.05}$ & $\sqrt[3]{25}$, we replace $x = 8.05$ & $x = 25$ respectively & find out $L(8.05)$ & $L(25)$

Now, $L(8.05) = 2 + \frac{1}{12}(8.05 - 8) = 2 + \frac{1}{12}(0.05)$

$= 2.00416667$

& $L(25) = 2 + \frac{1}{12}(25-8) = 2 + \frac{17}{12} = 3.41666667$

Note: $\sqrt[3]{8.05} = 2.00415862$ & $\sqrt[3]{25} = 2.92401774$



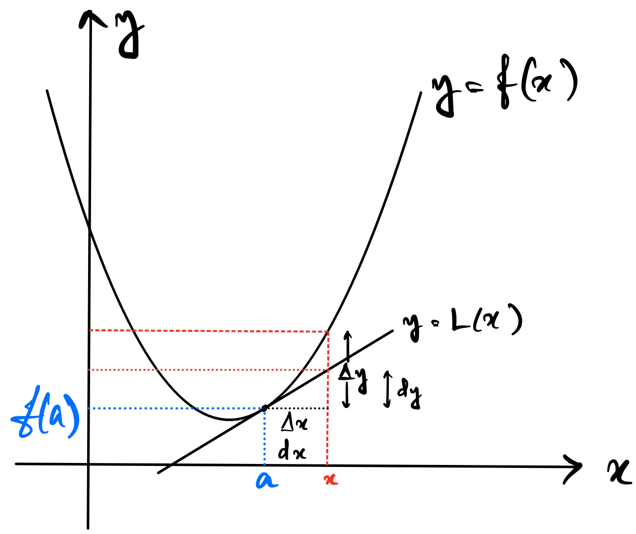
(*) <u>Remark</u>:- For $\sqrt[3]{25}$, if we find the linear approximation around $x=25$ (say $x=27$) then we will get more accurate result.

For instance: $L(x)$ about $x=27$ is $3+\frac{1}{27}(x-27)$

$$= 3+\frac{x}{27}-1$$

$$= 2+\frac{x}{27}.$$

& $L(25) = 2+\frac{25}{27} = 2.9259259$



Arbitrary Distance from a

$\Delta x = x - a$

$\Delta y = f(x) - f(a)$

$$\boxed{\Delta y = f(x+\Delta x) - f(x)}$$

Infinitesimal Distance from a.

$dx = x - a$

$dy = L(x) - L(a)$

$= [f(a) + f'(a)(x-a)] - f(a)$

$= f'(a)(x-a)$

$= f'(a)\,dx$

$$\boxed{dy = f'(x)\,dx} \longrightarrow \text{Differentials.}$$

Eg. Let $y = x^2 + 7$

(a) Find dy in terms of dx.

(b) Approximate Δy, if $x = 1$ & $\Delta x = 0.001$

$dy = f'(x)\,dx$ & $\Delta y = f(x+\Delta x) - f(x)$.

(a) $f(x) = x^2 + 7 \Rightarrow f'(x) = 2x$

So, $dy = 2x\,dx$.

(b) $\Delta y = f(1 + 0.001) - f(1)$

$= f(1.001) - f(1)$

$= [(1.001)^2 + 7] - [1^2 + 7]$

$= (1.001)^2 - 1^2 = (1.001 + 1)(1.001 - 1)$

$= (2.001)(0.001)$

$= 0.002001$